

7.5 Strategy for Integration

Strategy

1. Simplify the integrand if possible

- **Algebraic manipulation:** For example, $\int \sqrt{x}(1 + \sqrt{x}) \, dx = \int (x^{\frac{1}{2}} + x) \, dx$
- **Trigonometric identities:** For example,

$$\int (\sin x + \cos x)^2 \, dx = \int (\sin^2 x + 2 \sin x \cos x + \cos^2 x) \, dx = \int (1 + 2 \sin x \cos x) \, dx.$$

2. Look for an “obvious” substitution

- Try to find some function in the integrand whose differential also occurs, apart from a constant factor. For example, $\int \frac{x}{x^2 - 1} \, dx$, $u = x^2 - 1$ and $du = 2x \, dx$.
- This way we avoid the method of partial fractions.

3. Classify the integrand according to its form.

- (a) **Trigonometric functions:** If $f(x)$ is a product of powers of $\sin x / \cos x$, $\tan x / \sec x$, $\cot x / \csc x$ then use substitutions from section 7.2 and trigonometric identities.
- (b) **Rational functions:** If $f(x) = \frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials, then apply methods of partial fractions.
- (c) **Integration by parts:** If $f(x)$ is the product of a polynomial or a power of x and a trigonometric/exponential/logarithmic function, then try integration by parts. Choose u and dv according to the advice given in section 7.1
- (d) **Radicals:** If $\sqrt{\pm x^2 \pm a^2}$ occurs, use a trigonometric substitution according to the table in section 7.3.

4. If everything else fails...

- Manipulate the integrand. For example, $\int \frac{1}{1 - \cos x} \, dx = \int \frac{1}{1 - \cos x} \cdot \frac{1 + \cos x}{1 + \cos x} \, dx$
- Try a not so obvious substitution.

You might need the following integrals:

- $\int \sec x \, dx = \ln |\sec x + \tan x|$
- $\int \csc x \, dx = -\ln |\csc x + \cot x|$
- $\int \tan x \, dx = \ln |\sec x|$
- $\int \cot x \, dx = \ln |\sin x|$
- $\int \frac{1}{x^2 + a^2} \, dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right)$
- $\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \left(\frac{x}{a} \right), a > 0$

Example. Evaluate $\int \frac{x+2}{x^2+3x-4} dx$.

Partial fractions!

$$\left[\frac{x+2}{x^2+3x-4} = \frac{x+2}{(x+4)(x-1)} = \frac{A}{x+4} + \frac{B}{x-1} \right] (x+4)(x-1)$$

$$x+2 = A(x-1) + B(x+4)$$

$$\underline{x=1:} \quad 3 = B(5) \Rightarrow B = \frac{3}{5}$$

$$\underline{x=-4:} \quad -2 = A(-5) \Rightarrow A = \frac{2}{5}$$

$$\int \frac{x+2}{x^2+3x-4} dx = \frac{2}{5} \int \frac{1}{x+4} dx + \frac{3}{5} \int \frac{1}{x-1} dx$$

$$= \boxed{\frac{2}{5} \ln|x+4| + \frac{3}{5} \ln|x-1| + C}$$

Example. Evaluate $\int \frac{\sin x}{\cos^3 x} dx$.

$$u = \cos x$$

$$du = -\sin x dx$$

$$\int \frac{\sin x}{\cos^3 x} dx = \int \frac{1}{u^3} du = -\int u^{-3} du = -\frac{u^{-2}}{-2} + C = \frac{1}{2} \cdot \frac{1}{u^2} + C$$

$$= \boxed{\frac{1}{2} \cdot \frac{1}{\cos^2 x} + C} = \frac{1}{2} \sec^2 x + C$$

Example. Evaluate $\int \frac{\sin^3 x}{\cos x} dx$.

$$\begin{aligned}
 \int \frac{\sin^3 x}{\cos x} dx &= \int \frac{\sin^2 x \cdot \sin x}{\cos x} dx = \int \frac{(1 - \cos^2 x) \cdot \sin x}{\cos x} dx \\
 &= - \int \frac{1 - u^2}{u} du = - \int \frac{1}{u} - u du \\
 &= -\ln|u| + \frac{u^2}{2} + C \\
 &= \boxed{-\ln|\cos x| + \frac{\cos^2 x}{2} + C}
 \end{aligned}$$

$u = \cos x$
 $du = -\sin x dx$

Example. Evaluate $\int x \sec x \tan x dx$.

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Integration by parts!

$$u = x \quad dv = \sec x \tan x dx$$

$$du = dx \quad \swarrow \quad v = \int \sec x \tan x dx = \sec x$$

$$\begin{aligned}
 \int x \sec x \tan x dx &= x \sec x - \int \sec x dx \\
 &= \boxed{x \sec x - \ln|\sec x + \tan x| + C}
 \end{aligned}$$

Example. Evaluate $\int \frac{x^2}{\sqrt{1-x^2}} dx$.

Trig. Substitution

$$x = \sin \theta$$

$$dx = \cos \theta d\theta$$

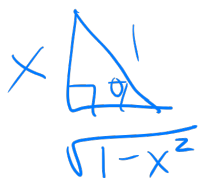
$$\int \frac{x^2}{\sqrt{1-x^2}} dx = \int \frac{\sin^2 \theta \cdot \cos \theta d\theta}{\sqrt{1-\sin^2 \theta}} = \int \frac{\sin^2 \theta \cdot \cos \theta}{\sqrt{\cos^2 \theta}} d\theta = \int \frac{\sin^2 \theta \cdot \cos \theta}{\cos \theta} d\theta$$

$$= \int \sin^2 \theta d\theta = \frac{1}{2} \int 1 - \cos 2\theta d\theta = \frac{1}{2} \left[\theta - \frac{\sin 2\theta}{2} \right] + C$$

$$\frac{1}{2} \sin 2\theta = \sin \theta \cos \theta \Rightarrow \sin 2\theta = 2 \sin \theta \cos \theta$$

$$x = \sin \theta$$

$$\sin^{-1} x = \theta$$



$$\frac{x}{\sqrt{1-x^2}}$$

$$= \frac{1}{2} \sin^{-1} x - \frac{1}{4} \cdot 2 \sin \theta \cos \theta + C$$

$$= \boxed{\frac{1}{2} \sin^{-1} x - \frac{1}{2} x \cdot \sqrt{1-x^2} + C}$$

Example. Evaluate $\int \frac{e^{\sqrt{t}}}{\sqrt{t}} dt$.

$$u = \sqrt{t}$$

$$du = \frac{1}{2\sqrt{t}} dt$$

$$2 du = \frac{1}{\sqrt{t}} dt$$

$$\int \frac{e^{\sqrt{t}}}{\sqrt{t}} dt = 2 \int e^u du = 2e^u + C$$

$$= \boxed{2e^{\sqrt{t}} + C}$$

Example. Evaluate $\int \tan^3 x \sec^4 x \, dx$.

$$\int \tan^3 x \sec^4 x \, dx = \int \tan^3 x \sec^2 x \cdot \sec^2 x \, dx = \int \tan^3 x (\tan^2 x + 1) \cdot \sec^2 x \, dx$$

$$u = \tan x$$

$$du = \sec^2 x \, dx$$

$$= \int u^3 (u^2 + 1) \, du = \int u^5 + u^3 \, du = \frac{u^6}{6} + \frac{u^4}{4} + C = \frac{\tan^6 x}{6} + \frac{\tan^4 x}{4} + C$$

$$\int \tan^3 x \sec^4 x \, dx = \int \tan^2 x \sec^3 x \cdot \sec x \tan x \, dx = \int (\sec^2 x - 1) \cdot \sec^3 x \cdot \sec x \tan x \, dx$$

$$u = \sec x$$

$$du = \sec x \tan x \, dx$$

$$= \int (u^2 - 1) u^3 \, du = \int u^5 - u^3 \, du = \frac{u^6}{6} - \frac{u^4}{4} + C$$

$$= \frac{\sec^6 x}{6} - \frac{\sec^4 x}{4} + C$$

Example. Evaluate $\int \frac{1}{x\sqrt{4x^2+1}} \, dx$.

$$2x = \tan \theta \Rightarrow x = \frac{1}{2} \tan \theta$$

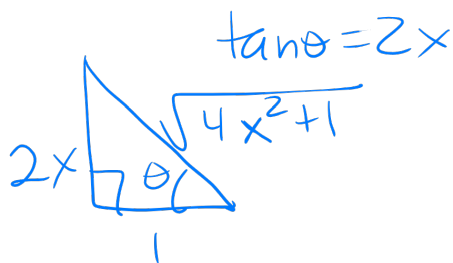
$$dx = \frac{1}{2} \sec^2 \theta \, d\theta$$

$$\int \frac{1}{x\sqrt{4x^2+1}} \, dx = \int \frac{\frac{1}{2} \sec^2 \theta \, d\theta}{\frac{1}{2} \tan \theta \sqrt{\tan^2 \theta + 1}} = \int \frac{\sec^2 \theta}{\tan \theta \sqrt{\sec^2 \theta}} \, d\theta$$

$$= \int \frac{\sec \theta}{\tan \theta} \, d\theta = \int \frac{\frac{1}{\cos \theta}}{\frac{\sin \theta}{\cos \theta}} \, d\theta = \int \frac{1}{\cos \theta} \cdot \frac{\cos \theta}{\sin \theta} \, d\theta$$

$$= \int \frac{1}{\sin \theta} \, d\theta = \int \csc \theta \, d\theta = -\ln |\csc \theta + \cot \theta| + C$$

$$= -\ln \left| \frac{\sqrt{4x^2+1}}{2x} + \frac{1}{2x} \right| + C$$



Example. Evaluate $\int \frac{2x-3}{x^3+3x} dx$.

$$\left[\frac{2x-3}{x^3+3x} = \frac{2x-3}{x(x^2+3)} = \frac{A}{x} + \frac{Bx+C}{x^2+3} \right] x(x^2+3)$$

$$\begin{aligned} 2x-3 &= A(x^2+3) + (Bx+C) \cdot x & 3A &= -3 \\ &= Ax^2+3A+Bx^2+Cx & A &= -1 \\ &= (A+B)x^2 + Cx + 3A & C &= 2 \end{aligned}$$

$$A+B=0 \Rightarrow B=-A=-(-1)=1$$

$$\int \frac{2x-3}{x^3+3x} dx = \int -\frac{1}{x} + \frac{x+2}{x^2+3} dx = -\int \frac{1}{x} dx + \int \frac{x}{x^2+3} dx + 2 \int \frac{1}{x^2+3} dx$$

$$\int \frac{1}{x^2+a^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right)$$

$$\begin{aligned} u &= x^2+3 \\ du &= 2x dx \\ \frac{1}{2} du &= x dx \end{aligned}$$

$$= -\ln|x| + \frac{1}{2} \int \frac{1}{u} du + 2 \cdot \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{x}{\sqrt{3}}\right)$$

$$= -\ln|x| + \frac{1}{2} \ln|u| + \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C$$

$$= \boxed{-\ln|x| + \frac{1}{2} \ln|x^2+3| + \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C}$$